

Classification and Application of Triangular Quark Mass Matrices

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Abstract

Any given pair of up and down quark mass matrices can be brought into the triangular form, at the same time eliminating their unphysical contents. Further, we show that every physically viable pair can always be reduced to one of ten triangular forms, which we list explicitly. When hermitian mass matrices are thus analyzed, it is found that any pattern of texture zeros translates into simple, analytic predictions for the quark mixings. Amongst possible hermitian matrices with five texture zeros, this method enables us to identify the unique pair which is favored by present data.

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1. Introduction.

In the study of quark mass matrices, one often starts from a presumed form and then derives relations between the masses and mixings that can be confronted with experimental data. Usually, the forms considered are hermitian with certain texture zeros [1,2], although non-hermitian matrices have also been considered [3]. It has recently been observed [4] that upper triangular matrices provide a most convenient basis to get the physical contents of general quark mass matrices. Through a rotation of the right-handed (RH) quarks, any mass matrix (defined as in $\overline{\psi}_L M \psi_R$) can be put into the upper triangular form, with three zero elements at the lower-left corner. When either the up or the down quark mass matrix is diagonal, the triangular mass matrix elements of the other consist simply of products of a mass and a Cabbibo-Kobayashi-Maskawa (CKM) matrix element. In particular, the diagonal elements are approximately equal to the quark masses, and the off-diagonal elements directly tell us about the mixing among the three families. CP -violating effects depend on one independent phase that can reside in any of the $(1, 2)$, $(1, 3)$, $(2, 2)$, and $(2, 3)$ positions of the triangular matrix. It is obvious that the total number of parameters in the mass matrices is equal to the number of physical observables, namely, the six quark masses, three mixing angles, and one CP phase. In fact, among the existing minimal parameter bases [5,6], where the unphysical degrees of freedom are eliminated, upper triangular mass matrices exhibit most clearly and simply the quark masses and CKM angles.

The purpose of this letter is two-fold. Firstly, we would like to extend our previous analysis to the more general case where neither the U - nor the D -quark mass matrix is diagonal. In particular, we classify all the upper triangular mass matrices in the minimal parameter basis – with nine nonzero elements distributed between the up and down quark mass matrices. The observed hierarchical structure of quark masses and small mixing angles allows us to choose the hierarchical basis for the mass matrices in which the $(3, 3)$ is the largest element and the rotation angles needed for diagonalization are all small. In this basis, we obtain a complete set of ten pairs of upper triangular quark mass matrices. To a good approximation, each entry is simply given by either a quark mass or its product with

one or two CKM elements. Any given pair of mass matrices can always be reduced to one of the pairs listed in Table I. In so doing, one can easily read off the physical masses and mixings implied by these matrices.

Secondly, we use triangular matrices to study texture zeros in hermitian mass matrices which may be a manifestation of certain flavor symmetries. It is found that simple relations can be established between the parameters of hierarchical hermitian and triangular matrices. By reducing generic hermitian matrices into one of the pairs in Table I, we show that any given pattern of texture zeros in the hermitian matrices implies simple relations between quark masses and mixing angles. Specifically, analyses of these relations show that only one of the five pairs of hermitian matrices studied by Ramond-Roberts-Ross (RRR) in Ref. [2] is favored by current data.

2. Classification.

We start by deriving the triangular form of the down (up) quark mass matrix in the basis where M^U (M^D) is diagonal. Then we generate all the other triangular textures in the minimal parameter basis through a common left-handed (LH) rotation on both M^U and M^D together with separate RH rotations, when necessary. Without loss of generality, we will work in a basis where $\det V_{\text{CKM}} = +1$, which will simplify the expressions for the mass matrices.

For $M^U = \text{Diag}(m_u, m_c, m_t)$, M^D is given by $M^D = V_{\text{CKM}} \text{Diag}(m_d, m_s, m_b)$ up to a RH unitary rotation. By applying to M^D three successive RH rotations, we can bring the $(3, 1)$, $(3, 2)$, and $(2, 1)$ elements of M^D to zero and arrive at the upper triangular form [4],

$$M^D = \begin{pmatrix} \frac{m_d}{V_{ud}^*} & m_s V_{us} & m_b V_{ub} \\ 0 & m_s V_{cs} & m_b V_{cb} \\ 0 & 0 & m_b V_{tb} \end{pmatrix} \times (1 + \mathcal{O}(\lambda^4)) , \quad (M^U \text{ diagonal}) \quad (1)$$

where $\lambda = |V_{us}| = 0.22$ [7].

Eq. (1) is almost the same as $V_{\text{CKM}} \text{Diag}(m_d, m_s, m_b)$ by striking out the three lower-left matrix elements. The reason for this simple form is because the afore-mentioned RH

rotations are all very small, being of order λ^7 , λ^4 , and λ^3 for the (31), (32), and (21) rotations, respectively. The most noticeable effect of these comes from the (21) rotation which changes the (1, 1) element of M^D by $1 + \mathcal{O}(\lambda^2)$. The factor $1/V_{ud}^*$ arises because of the invariance of the determinant, and because $V_{ud}^* = V_{cs}V_{tb} - V_{cb}V_{ts} \simeq V_{cs}V_{tb}$, for $\det V_{\text{CKM}} = +1$. Note that, by transforming a hierarchical matrix into the upper triangular form, one has eliminated the RH rotations necessary for its diagonalization.

By contrast, lower triangular mass matrices [6], with zeros in the upper-right corner, do not provide a simple relation between its matrix elements and the quark masses and CKM angles. The reason is that, had we tried to make the matrix $M^D = V_{\text{CKM}} \text{Diag}(m_d, m_s, m_b)$ lower triangular, we would have to use large angle RH rotations. In doing so, the resulting lower triangular matrix elements are very different from those of the original matrix, and no simple relations are expected.

When M^D is diagonal, the upper triangular form for M^U can be simply obtained from Eq. (1) by replacing V_{ij} with V_{ji}^* [4],

$$M^U = \begin{pmatrix} \frac{m_u}{V_{ud}} & m_c V_{cd}^* & m_t V_{td}^* \\ 0 & m_c V_{cs}^* & m_t V_{ts}^* \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix} \times (1 + \mathcal{O}(\lambda^4)) . \quad (M^D \text{ diagonal}) \quad (2)$$

Eqs. (1) and (2) will serve as our starting point for getting all the other triangular mass matrices with nine nonzero elements. To retain a simple expression for them in terms of physical parameters, we will work in the hierarchical basis in which the magnitudes of the matrix elements, if non-zero, satisfy $M_{33} \gg M_{22} \gg M_{11}$, $M_{33} \gg M_{23} \gg M_{13}$ and $M_{22} \gg M_{12}$.

To generate other triangular textures from Eq. (1), we can set one of the off-diagonal elements in M^D to zero through a LH D -quark rotation (more precisely, this is the leading order term of a succession of LH and RH rotations with decreasing angles). The same LH rotation acting on M^U then brings the diagonal M^U into upper triangular form with one non-zero off-diagonal element. To remain in the hierarchical basis, only small angle (smaller

than $\pi/4$) LH rotations are allowed. We thus obtain the first five pairs of matrices as listed in Table I. From Eq. (2), another set of five pairs of triangular matrices can be obtained. They are related to the first five in Table I by the operation: $M^U \leftrightarrow M^D$ and $V_{ij} \leftrightarrow V_{ji}^*$. Note that the texture zeros are phenomenological zeros, with negligible but nonvanishing physical effects. In this way, we arrive at a complete list of ten pairs of upper triangular textures in the hierarchical, minimal parameter basis.

For the textures of Table I, the CKM matrix can be written, to a good approximation, as a product of three LH rotations, coming separately from the diagonalization of M^U and M^D . The rotation angles are approximately given by the ratios of CKM elements. These are also listed in Table I. Finally, for each pair of textures in the minimal parameter basis, there is only one physical phase which can be written as a linear combination of either four or six phases of the matrix elements. This phase is simply related to one of the three angles of the unitarity triangle: $\alpha \equiv \arg \left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*} \right]$, $\beta \equiv \arg \left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*} \right]$, and $\gamma \equiv \arg \left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \right]$. It enters into the CP -violating Jarlskog parameter [8], the approximate form of which is also listed in Table I.

The simplicity of the triangular mass matrices in Table I is evident: each matrix element is either a quark mass or its product with one or two CKM elements. By contrast, other minimal-parameter textures cannot be written in such a simple form. Table I allows one to read off immediately the physical masses and mixings of any quark mass matrices after casting them into one of the ten triangular patterns. The conversion can be achieved through separate RH rotations and possibly LH rotations which are common to both M^U and M^D . This could avoid the sometimes tedious process of matrix diagonalization [4].

3. Triangular versus hermitian mass matrices.

One can equally well start from upper triangular mass matrices, in which the unphysical RH rotations are eliminated, and generate other forms of mass matrices by appropriate RH rotations. In particular, we may generate hermitian mass matrices, which have been the focus of many studies. Here we illustrate the method by examining analytically hermitian mass matrices with five texture zeros, as was studied in [2].

Using the mass relations $m_u : m_c : m_t \sim \lambda^8 : \lambda^4 : 1$ and $m_d : m_s : m_b \sim \lambda^4 : \lambda^2 : 1$ evaluated at the weak scale, and the CKM rotations $V_{us} = \lambda$, $V_{cb} \sim \lambda^2$, $V_{ub} \sim \lambda^4$, and $V_{td} \sim \lambda^3$, we can write the properly normalized Yukawa matrices for U and D in the most general triangular form,

$$T^U = \begin{pmatrix} a_U \lambda^8 & b_U \lambda^6 & c_U \lambda^4 \\ 0 & d_U \lambda^4 & e_U \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^D = \begin{pmatrix} a_D \lambda^4 & b_D \lambda^3 & c_D \lambda^3 \\ 0 & d_D \lambda^2 & e_D \lambda^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

where the diagonal coefficients are determined by quark masses and are of order one, and the off-diagonal coefficients can be either of order one or much smaller. Without loss of generality, we can take the diagonal elements to be real in the following analysis. In writing the above triangular matrices, we have required the LH (12), (23), and (13) rotations for diagonalizing each sector to be respectively of order V_{us} , V_{cb} , and V_{td} (V_{ub}) or smaller¹. This is known as the naturalness criteria [9], and can always be implemented by a common LH rotation to both T^U and T^D .

From the hierarchical form of Eq. (3), one can directly read off both the mass eigenvalues and the LH unitary matrices that diagonalize T^U and T^D . The quark masses are simply related to the diagonal elements,

$$(m_u, m_c) = m_t(a_U \lambda^8, d_U \lambda^4) \times (1 + \mathcal{O}(\lambda^4)), \quad (4)$$

$$(m_d, m_s) = m_b(a_D \lambda^4, d_D \lambda^2) \times (1 + \mathcal{O}(\lambda^2)). \quad (5)$$

The CKM matrix is obtained from separate LH diagonalizing rotations for T^U and T^D [4],

$$\begin{aligned} V_{\text{CKM}} &= V_U^\dagger V_D, \\ V_U &\simeq R_{23}(e_U \lambda^2) R_{13}(c_U \lambda^4) R_{12}(b_U \lambda^2 / d_U), \\ V_D &\simeq R_{23}(e_D \lambda^2) R_{13}(c_D \lambda^3) R_{12}(b_D \lambda / d_D). \end{aligned} \quad (6)$$

¹Though we take the LH (12) rotation in the U sector to be of order $\sqrt{m_u/m_c} \sim \lambda^2$ for convenience, one can also start with $T_{12}^U = b_U \lambda^5$ and arrive at the same results.

The mass matrices can take different but physically equivalent forms through RH rotations. In particular, they can be transformed into the following hermitian form ($T^i \rightarrow Y^i \simeq T^i V_i^\dagger$, $i = U, D$),

$$Y^U = \begin{pmatrix} \left(a_U + c_U c_U^* + \frac{b_U b_U^*}{d_U}\right) \lambda^8 & (b_U + c_U e_U^*) \lambda^6 & c_U \lambda^4 \\ (b_U^* + c_U^* e_U) \lambda^6 & (d_U + e_U e_U^*) \lambda^4 & e_U \lambda^2 \\ c_U^* \lambda^4 & e_U^* \lambda^2 & 1 \end{pmatrix} \times (1 + \mathcal{O}(\lambda^4)) , \quad (7)$$

$$Y^D = \begin{pmatrix} \left(a_D + \frac{b_D b_D^*}{d_D}\right) \lambda^4 & b_D \lambda^3 & c_D \lambda^3 \\ b_D^* \lambda^3 & d_D \lambda^2 & e_D \lambda^2 \\ c_D^* \lambda^3 & e_D^* \lambda^2 & 1 \end{pmatrix} \times (1 + \mathcal{O}(\lambda^2)) . \quad (8)$$

A few interesting observations can now be made about Y^U and Y^D .

1. *hermitian vs. triangular zeros*: Except for Y_{12}^U , it is seen that an off-diagonal zero² in the triangular form has a one to one correspondence to that in the hermitian form. On the other hand, whereas diagonal zeros are not allowed in the triangular form (corresponding to vanishing quark masses), for Y^U and Y^D such zeros imply definite relations among elements of the triangular matrix.
2. *Y^U vs. Y^D zeros*: A notable difference between Y^U and Y^D is that Y_{22}^U can be zero, but $Y_{22}^D \neq 0$ ³. $Y_{11}^D = 0$ implies that the (12) rotation angle for Y^D is $\theta_{12}^D \simeq \sqrt{m_d/m_s}$. The condition $Y_{22}^U = 0$ implies $\theta_{23}^U \simeq \sqrt{m_c/m_t}$. On the other hand, $\theta_{12}^U \simeq \sqrt{m_u/m_c}$ can be obtained if both $Y_{11}^U = 0$ and $Y_{13}^U = 0$ (more precisely $|c_U^2/a_U| \ll 1$) are satisfied. One other relation, $\theta_{23}^U \theta_{13}^U / \theta_{12}^U \simeq m_c/m_t$, can be obtained if $Y_{12}^U = 0$.

²Only zeros along and above the diagonals of triangular and hermitian matrices are counted.

³More precisely, $Y_{22}^D = (d_D + |e_D|^2 \lambda^2) \lambda^2$ so that $Y_{22}^D = 0$ would imply $\theta_{23}^D \simeq \sqrt{m_s/m_b} \sim \lambda$, which is too large to be partially canceled by $\sqrt{m_c/m_t}$ to get $V_{cb} \sim \lambda^2$. Had the cancellation somehow occurred, it would involve $\mathcal{O}(\lambda)$ fine tuning which is against the naturalness criteria.

3. Y_{11}^D and Y_{12}^D cannot be both equal to zero. One can therefore have at most 3 texture zeros for Y^D corresponding to $Y_{11}^D = Y_{13}^D = Y_{23}^D = 0$ or $Y^D = \text{Diag}(m_d/m_b, m_s/m_b, 1)$. Similarly, Y_{22}^U and Y_{23}^U cannot be both zero, and one can easily show that at most 3 texture zeros are allowed for Y^U with five possible forms including the diagonal matrix.
4. The zeros in Y^U and Y^D , which imply relations between the elements of the triangular matrices, translate into relations between physical parameters by referring to Table I. We can thus quickly rule out pairs of Y^U and Y^D 's which are obviously untenable. It is easy to check that there are no viable pairs with six texture-zeros, and that only five pairs with five texture zeros are possible candidates, in agreement with the findings of Ref. [2]. However, the triangular matrix method also allows us to investigate each pair analytically, and to have a more in-depth evaluation of the physical implications of these matrices.

In Table II, we list five types of hermitian matrices, together with their corresponding triangular forms. These five hermitian matrices can be paired up to give the five RRR patterns for the quark Yukawa matrices, (Y^U, Y^D) , which are listed in Table III. Using Tables II and III, we see that RRR patterns 1, 3, and 4 simply correspond to triangular textures II, III, and VII of Table I, respectively. The triangular form of RRR pattern 2, (M_2, M_4) , has only two texture zeros, and an appropriate LH (23) rotation can be used to generate a third zero and transform it into either triangular patterns II or VII. Similarly, a LH (12) rotation is necessary to bring RRR pattern 5, (M_5, M_1) , into the minimal-parameter triangular patterns VI, VII, or X. The physical implications are summarized in the last column of Table III. For instance, the relation $|b_U|^2 = -a_U d_U$ for pattern 1, referring to texture II of Table I, reads $|m_c \frac{V_{ub}}{V_{cb}}|^2 = |m_u m_c|$, or $|\frac{V_{ub}}{V_{cb}}|^2 = |\frac{m_u}{m_c}|$. Each of the five RRR patterns entails two physical predictions⁴. We turn now to a detailed analysis of these

⁴For patterns 2 and 5, because of the necessary LH rotation, the three relations listed also give rise to only two physical predictions.

predictions.

From Tables I and III, RRR patterns 1, 2, and 4 give rise to the same predictions:

$$\left| \frac{V_{ub}}{V_{cb}} \right| = \sqrt{\frac{m_u}{m_c}} \times (1 + \mathcal{O}(\lambda^3)) = 0.059 \pm 0.006 , \quad (9)$$

$$\left| \frac{V_{td}}{V_{ts}} \right| = \sqrt{\frac{m_d}{m_s}} \times (1 + \mathcal{O}(\lambda^2)) = 0.224 \pm 0.022 , \quad (10)$$

where the quark masses are taken from Ref. [11]. It is interesting to note that the ratios of the quark masses and mixings in Eqs. (9) and (10) are all scale-independent [10,2]. Furthermore, Eqs. (9,10) are independent of the phases in the mass matrices. Experimentally,

$$\left| \frac{V_{ub}}{V_{cb}} \right|_{\text{exp}} = 0.093 \pm 0.014 , \quad (11)$$

$$0.15 < \left| \frac{V_{td}}{V_{ts}} \right|_{\text{exp}} < 0.24 , \quad (12)$$

where V_{ub} is from a recent measurement [12,13] and $\left| \frac{V_{td}}{V_{ts}} \right|$ comes from a 95% C.L. standard model fit to electroweak data [13]. We may conclude that the 1st, 2nd, and 4th RRR patterns are disfavored by the data on V_{ub}/V_{cb} .

Tables I and III also yield two predictions for the 3rd RRR pattern⁵:

$$|V_{ub}| = \sqrt{\frac{m_u}{m_t}} \times (1 + \mathcal{O}(\lambda^2)) = 0.0036 \pm 0.0004 , \quad (13)$$

$$\left| \frac{V_{us}}{V_{cs}} \right| = \sqrt{\frac{m_d}{m_s}} \times (1 + \mathcal{O}(\lambda^2)) = 0.224 \pm 0.022 . \quad (14)$$

While Eq. (14) is scale-independent, Eq. (13) is scale-dependent, and the number given there is at M_Z . In the framework of supersymmetric grand unified theories (SUSY GUT), pattern 3 and thus Eqs. (13) and (14) are assumed to be valid at $M_X = 2 \times 10^{16}$ GeV. Renormalization group equations (RGE) running gives V_{ub} a central value ≈ 0.0033 at M_Z . The predictions are in good agreement with present data, approximately independent of the scale at which the pattern is valid.

⁵A recent study of this pattern was done in [14]

The 5th RRR pattern differs from the other four in that it allows for two separate solutions to the Yukawa matrices with different predictions for quark mixings. These two solutions correspond to $d_U > 0$ (5a) and $d_U < 0$ (5b). Pattern 5 can be brought into texture VII of Table I by a common LH (12) rotation $R_{12}^\dagger(c_U \lambda^2/e_U)$. The two predictions $a_U = -|c_U|^2 \left(1 + \frac{|e_U|^2}{d_U}\right)$ and $|b_D|^2 = -a_D d_D$ can be both written in terms of physical parameters. After some manipulations, we obtain

$$\left|\frac{m_u}{m_t}\right| = |V_{ub}|^2 \Gamma_\pm \times (1 + \mathcal{O}(\lambda^2)) , \quad (15)$$

$$\left|\frac{m_d}{m_s}\right| = \left|\frac{1}{V_{ts}^*}\right|^2 \left|\frac{V_{ub}}{V_{cs}^*} \Gamma_\pm + V_{td}^*\right|^2 \times (1 + \mathcal{O}(\lambda^2)) , \quad (16)$$

$$\Gamma_\pm^{-1} \equiv 1 \pm \left|\frac{m_t}{m_c}\right| \frac{|V_{ts}^*|^2}{|V_{cs}^*|} . \quad (17)$$

Here the \pm sign in Γ_\pm corresponds to choosing the \pm sign for d_U . Note that Γ_\pm is scale-dependent. But it can be verified that Eq. (16) is rephasing invariant. Numerically, $|\Gamma_\pm^{-1} - 1| \simeq 0.43$ at M_Z .

It is interesting to note that in the limit $\Gamma_\pm \rightarrow 1$, Eqs. (15,16) are reduced to the two predictions of the 3rd RRR pattern (Eqs. (13,14)): $m_u/m_t \simeq |V_{ub}|^2$ and $m_d/m_s \simeq |V_{cd}|^2$ (using $V_{ub} + V_{cs}^* V_{td}^* = V_{cd}^* V_{ts}^*$). The significant deviation of Γ_\pm from unity shows that the predictions of the 3rd and the 5th RRR patterns are, in principle, mutually exclusive. To obtain a numerical estimate, Eq. (15) can be rewritten as (including $\mathcal{O}(\lambda^2)$ terms),

$$\left|\frac{V_{ub}}{V_{cb}}\right| = \sqrt{\frac{m_u}{m_c} \left(\frac{m_c}{m_t V_{cb}^2} \pm 1\right)} \times (1 + \mathcal{O}(\lambda^3)) = \begin{cases} 0.107 \pm 0.012 & (d_U > 0, \text{ 5a}) \\ 0.068 \pm 0.011 & (d_U < 0, \text{ 5b}) \end{cases} \quad (18)$$

where the numbers are given at M_Z with $V_{cb} = 0.040$. If pattern 5 is assumed to be valid at M_X as in SUSY GUT, the central values of $V_{ub}/V_{cb} \approx 0.103$ ($d_U > 0$) and ≈ 0.062 ($d_U < 0$) for $V_{cb} \simeq 0.033$ at M_X . Indeed, we see that the V_{ub}/V_{cb} predictions of patterns 3 and 5 are significantly different. Future measurements of V_{ub} at the B factories should be able to distinguish between the two patterns. Note that Eq. (16) is sensitive to the relative phase between V_{ub} and V_{td}^* which is fixed by fitting $|V_{us}|$, and a simple estimate of V_{td}/V_{ts} cannot

be similarly made. However, the fact that Γ_{\pm} is very different from unity, together with Eqs.(14,16), causes V_{td}/V_{ts} to deviate considerably from its standard model best-fit value.

To check the analytic predictions, we have solved numerically the five hermitian RRR patterns using the quark masses [11] and $V_{cb} = 0.040$ as input parameters. For RRR patterns 1, 2, 4, and 5, we fix the phase to get the correct value of V_{us} . V_{ub} and V_{td} are then obtained as outputs. For pattern 3, the phase is fixed by V_{td} since V_{us} as predicted in Eq. (14) is insensitive to it. Here V_{ub} and V_{us} are outputs. The results for the CKM elements at M_Z are given in Table IV. The two sets of numbers correspond to assuming the patterns to be valid at M_Z (M_X). The predictions of Eqs. (9, 10, 13, 14, 18) are in good agreement with the exact numerical results. Based on the values of V_{td} and V_{ub} given in Table IV, it is seen that pattern 5a is marginal and 5b is not favored. Note that the accuracy of the analytic predictions can be systematically improved by including higher order terms in λ , assuming the quark masses are precisely known.

To summarize, RRR patterns 1, 2, and 4 give a too small V_{ub}/V_{cb} , pattern 5b gives a too large V_{td}/V_{ts} and a small V_{ub}/V_{cb} , the fitting of pattern 5a with data requires a bit of stretch, and only the 3rd RRR pattern fits the data well. This conclusion is in disagreement with the results of Ref. [15].

4. Conclusions.

Generic mass matrices contain not only the physical parameters (masses and mixing angles), but also arbitrary right-handed rotations as well as common left-handed rotations for both the U - and D -quark sectors. These rotations can mask the real features of the mass matrices. In the minimal parameter scheme, one fixes these rotations by imposing a sufficient number of conditions on the mass matrices. A realization of this scheme is to transform both the U - and D -type matrices into the upper triangular form, and demand that there be three texture zeros shared by the two matrices. Because of the hierarchical structures in both the quark masses and their mixing angles, the resulting matrices are particularly simple in the hierarchical basis. All of their matrix elements are simple products of the masses and the CKM matrix elements. A complete classification of these matrices has been given and is

listed in Table I.

Triangular matrices can be easily put into hermitian form and used to classify and analyze hermitian texture zeros. Diagonal zeros in the hermitian mass matrices often manifest as simple relations between triangular matrix elements, which in turn imply certain relations between quark mixing angles and masses. By establishing the connection between triangular and hermitian matrices, we have analyzed hermitian mass matrices with five texture zeros. Simple, analytic predictions for quark mixings are obtained for each of the five Ramond-Roberts-Ross patterns. In particular, we note that the 5th RRR pattern allows two distinctive predictions, and that patterns 3 and 5 are mutually exclusive. Comparison with data indicates that patterns 1, 2, and 4 are disfavored by the combined measurements of V_{ub}/V_{cb} , that pattern 5 is marginally acceptable, and that only pattern 3 fits well.

Triangular matrices also allow one to go beyond hermitian mass matrices and study texture zeros in generic mass matrices which may be associated with certain flavor symmetries. In fact, any conceivable relations between quark masses and mixings translate simply into certain relations among the triangular matrix parameters. A transformation of the triangular matrices to matrices in other forms may reveal texture zeros in the new basis. This method can be both useful for analyzing general mass matrices, and helpful in a model-independent, bottom-up approach to the problem of quark masses and mixings.

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TABLES

TABLE I. Classification of triangular quark mass matrices in the hierarchical, minimal parameter basis. The form of each pattern is invariant under phase redefinitions of the quarks that satisfy $\det V_{\text{CKM}} = 1$. The CP -violating measure, $J.T.B. \simeq J m_t^4 m_b^4 m_c^2 m_s^2$, with J being the Jarlskog parameter; $A_i \equiv |M_{1i}^U|$ ($i = 1, 2, 3$), $B_1 \equiv |M_{22}^U|$, $B_2 \equiv |M_{23}^U|$, and $C \equiv |M_{33}^U|$, while $a_i \equiv |M_{1i}^D|$ ($i = 1, 2, 3$), $b_1 \equiv |M_{22}^D|$, $b_2 \equiv |M_{23}^D|$, and $c \equiv |M_{33}^D|$. α , β , and γ are angles of the unitarity triangle. The mass matrix elements of patterns I, III, IV, VI, VIII, IX are accurate up to $1 + \mathcal{O}(\lambda^4)$ corrections, those of II are corrected by $1 + \mathcal{O}(\lambda^3)$, and those of V, VII, X by $1 + \mathcal{O}(\lambda^2)$.

Texture	M^U	M^D	V_{CKM}	$J.T.B$
I	$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$	$\begin{pmatrix} \frac{m_d}{V_{ud}^*} & m_s V_{us} & m_b V_{ub} \\ 0 & m_s V_{cs} & m_b V_{cb} \\ 0 & 0 & m_b V_{tb} \end{pmatrix}$	$R_{23}^D R_{13}^D R_{12}^D$	$a_2 a_3 b_1 b_2 c^2 B_1^2 C^4 \sin \Phi_I$ ($\Phi_I = \arg\left(\frac{a_2 b_2}{a_3 b_1}\right) \simeq \gamma$)
II	$\begin{pmatrix} m_u & -m_c \frac{V_{ub}}{V_{cb}} & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$	$\begin{pmatrix} \frac{m_d}{V_{ud}^*} & m_s \frac{V_{td}^*}{V_{cb}} & 0 \\ 0 & m_s V_{cs} & m_b V_{cb} \\ 0 & 0 & m_b V_{tb} \end{pmatrix}$	$R_{12}^U R_{23}^D R_{12}^D$	$a_2 b_1 b_2^2 c^2 A_2 B_1 C^4 \sin \Phi_{II}$ ($\Phi_{II} = \arg\left(\frac{A_2 b_1}{a_2 B_1}\right) \simeq \alpha$)
III	$\begin{pmatrix} m_u & 0 & -m_t \frac{V_{ub}}{V_{tb}} \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$	$\begin{pmatrix} \frac{m_d}{V_{ud}^*} & m_s V_{us} & 0 \\ 0 & m_s V_{cs} & m_b V_{cb} \\ 0 & 0 & m_b V_{tb} \end{pmatrix}$	$R_{13}^U R_{23}^D R_{12}^D$	$a_2 b_1 b_2 c^3 A_3 B_1^2 C^3 \sin \Phi_{III}$ ($\Phi_{III} = \arg\left(\frac{A_3 b_1 c}{a_2 b_2 C}\right) \simeq \pi - \gamma$)
IV	$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & -m_t \frac{V_{cb}}{V_{tb}} \\ 0 & 0 & m_t \end{pmatrix}$	$\begin{pmatrix} \frac{m_d}{V_{ud}^*} & m_s V_{us} & m_b V_{ub} \\ 0 & m_s V_{cs} & 0 \\ 0 & 0 & m_b V_{tb} \end{pmatrix}$	$R_{23}^U R_{13}^D R_{12}^D$	$a_2 a_3 b_1 c^3 B_1^2 B_2 C^3 \sin \Phi_{IV}$ ($\Phi_{IV} = \arg\left(\frac{a_3 b_1 C}{a_2 c B_2}\right) \simeq \pi - \gamma$)
V	$\begin{pmatrix} m_u & -m_c \frac{V_{us}}{V_{cs}} & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$	$\begin{pmatrix} \frac{m_d}{V_{ud}^*} & 0 & -m_b \frac{V_{td}^*}{V_{cs}} \\ 0 & m_s V_{cs} & m_b V_{cb} \\ 0 & 0 & m_b V_{tb} \end{pmatrix}$	$R_{12}^U R_{23}^D R_{13}^D$	$a_3 b_1^2 b_2 c^2 A_2 B_1 C^4 \sin \Phi_V$ ($\Phi_V = \arg\left(\frac{a_3 B_1}{b_2 A_2}\right) \simeq \beta$)

Texture	M^U	M^D	V_{CKM}	$J.T.B$
VI	$\begin{pmatrix} \frac{m_u}{V_{ud}} & m_c V_{cd}^* & m_t V_{td}^* \\ 0 & m_c V_{cs}^* & m_t V_{ts}^* \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix}$	$\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$	$R_{12}^U R_{13}^U R_{23}^U$	$A_2 A_3 B_1 B_2 C^2 b_1^2 c^4 \sin \Phi_{\text{VI}}$ $(\Phi_{\text{VI}} = \arg\left(\frac{A_3 B_1}{A_2 B_2}\right) \simeq \beta)$
VII	$\begin{pmatrix} \frac{m_u}{V_{ud}} & m_c \frac{V_{ub}}{V_{ts}^*} & 0 \\ 0 & m_c V_{cs}^* & m_t V_{ts}^* \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix}$	$\begin{pmatrix} m_d & -m_s \frac{V_{td}^*}{V_{ts}^*} & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$	$R_{12}^U R_{23}^U R_{12}^D$	$A_2 B_1 B_2^2 C^2 a_2 b_1 c^4 \sin \Phi_{\text{VII}}$ $(\Phi_{\text{VII}} = \arg\left(\frac{A_2 b_1}{B_1 a_2}\right) \simeq \alpha)$
VIII	$\begin{pmatrix} \frac{m_u}{V_{ud}} & m_c V_{cd}^* & 0 \\ 0 & m_c V_{cs}^* & m_t V_{ts}^* \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix}$	$\begin{pmatrix} m_d & 0 & -m_b \frac{V_{td}^*}{V_{tb}^*} \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$	$R_{12}^U R_{23}^U R_{13}^D$	$A_2 B_1 B_2 C^3 a_3 b_1^2 c^3 \sin \Phi_{\text{VIII}}$ $(\Phi_{\text{VIII}} = \arg\left(\frac{A_2 B_2 c}{B_1 C a_3}\right) \simeq \pi - \beta)$
IX	$\begin{pmatrix} \frac{m_u}{V_{ud}} & m_c V_{cd}^* & m_t V_{td}^* \\ 0 & m_c V_{cs}^* & 0 \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix}$	$\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & -m_b \frac{V_{ts}^*}{V_{tb}^*} \\ 0 & 0 & m_b \end{pmatrix}$	$R_{12}^U R_{13}^U R_{23}^D$	$A_2 A_3 B_1 C^3 b_1^2 b_2 c^3 \sin \Phi_{\text{IX}}$ $(\Phi_{\text{IX}} = \arg\left(\frac{A_2 C b_2}{A_3 B_1 c}\right) \simeq \pi - \beta)$
X	$\begin{pmatrix} \frac{m_u}{V_{ud}} & 0 & -m_t \frac{V_{ub}}{V_{cs}^*} \\ 0 & m_c V_{cs}^* & m_t V_{ts}^* \\ 0 & 0 & m_t V_{tb}^* \end{pmatrix}$	$\begin{pmatrix} m_d & -m_s \frac{V_{cd}^*}{V_{cs}^*} & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$	$R_{13}^U R_{23}^U R_{12}^D$	$A_3 B_1^2 B_2 C^2 a_2 b_1 c^4 \sin \Phi_{\text{X}}$ $(\Phi_{\text{X}} = \arg\left(\frac{B_2 a_2}{A_3 b_1}\right) \simeq \gamma)$

TABLE II. The five hermitian matrices appearing in RRR patterns and their triangular forms. A hierarchical matrix structure as exhibited in Eqs. (3), (7), and (8) is assumed. $A' \equiv -|C|^2 \left(1 + \frac{|E|^2}{D}\right)$, $D' \equiv D + |E|^2$.

M_1	M_2	M_3	M_4	M_5
$\begin{pmatrix} 0 & B & 0 \\ B^* & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & 0 \\ B^* & 0 & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & C \\ 0 & D & 0 \\ C^* & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & B & 0 \\ B^* & D' & E \\ 0 & E^* & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & C \\ 0 & D' & E \\ C^* & E^* & 1 \end{pmatrix}$
$\begin{pmatrix} -\frac{ B ^2}{D} & B & 0 \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{ B ^2}{ E ^2} & B & 0 \\ 0 & - E ^2 & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} - C ^2 & 0 & C \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{ B ^2}{D} & B & 0 \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} A' & -CE^* & C \\ 0 & D & E \\ 0 & 0 & 1 \end{pmatrix}$

TABLE III. The five RRR hermitian patterns and their corresponding triangular patterns of Table I. Also shown are the relations among the triangular matrix elements implied by each pattern. For M_i ($i = 1, \dots, 5$), see Table II. a_j, \dots, e_j ($j = U, D$) are defined in Eq. (3).

RRR Pattern	Y^U	Y^D	Triangular Pattern	Relations
1	M_1	M_4	II	$ b_U ^2 = -a_U d_U$ $ b_D ^2 = -a_D d_D$
2	M_2	M_4	II or VII (LH 23 rotation)	$ b_U ^2 = -a_U d_U$ $ e_U ^2 = -d_U$ $ b_D ^2 = -a_D d_D$
3	M_3	M_4	III	$ c_U ^2 = -a_U$ $ b_D ^2 = -a_D d_D$
4	M_4	M_1	VII	$ b_U ^2 = -a_U d_U$ $ b_D ^2 = -a_D d_D$
5	M_5	M_1	VI, VII, or X (LH 12 rotation)	$b_U = -c_U e_U^*$ $ c_U ^2 = -a_U / \left(1 + \frac{ e_U ^2}{d_U}\right)$ $ b_D ^2 = -a_D d_D$

TABLE IV. Numerical solutions to the five RRR patterns. Inputs are the central values of quark masses and V_{cb} and V_{us} for patterns 1, 2, 4, and 5, and V_{cb} and V_{us} for pattern 3. Cases 5a and 5b refer to $d_U > 0$ and $d_U < 0$, respectively. The two sets of numbers correspond to assuming the patterns to be valid at M_Z ($M_X = 2 \times 10^{16}$ GeV of SUSY GUT). The CKM elements and J shown are at M_Z .

Pattern	V_{ud}	V_{us}	V_{ts}	$\left \frac{V_{ub}}{V_{cb}} \right $	$\left \frac{V_{td}}{V_{ts}} \right $	J
1	0.9755	0.220	0.0390	0.0583	0.217	1.9×10^{-5}
	(0.9755)	(0.220)	(0.0391)	(0.0584)	(0.217)	(1.9×10^{-5})
2	0.9753	0.221	0.0393	0.0592	0.207	1.9×10^{-5}
	(0.9754)	(0.221)	(0.0396)	(0.0589)	(0.209)	(1.9×10^{-5})
3	0.9758	0.219	0.0393	0.0934	0.193	2.7×10^{-5}
	(0.9757)	(0.219)	(0.0394)	(0.0871)	(0.193)	(2.6×10^{-5})
4	0.9755	0.220	0.0393	0.0585	0.224	2.0×10^{-5}
	(0.9755)	(0.220)	(0.0387)	(0.0585)	(0.223)	(1.9×10^{-5})
5a	0.9755	0.220	0.0393	0.107	0.238	3.7×10^{-5}
	(0.9754)	(0.220)	(0.0387)	(0.103)	(0.236)	(3.4×10^{-5})
5b	0.9754	0.221	0.0391	0.0670	0.245	2.3×10^{-5}
	(0.9753)	(0.221)	(0.0386)	(0.0607)	(0.243)	(2.0×10^{-5})